The Logic of Relations, Logical Substitution Groups, and Cardinal Numbers.

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PREFACE.

In Section I, the theory of logical equations is generalized; any definite logical equation is proved to correspond to a definite class of relations [cf. ★1·1 and ★1·2], and to each relation of the class corresponds a solution of the equation. But from the relational point of view the theory is equally simple, whether the number of variables in the corresponding logical equation is finite or infinite. Accordingly, we obtain a theory of logical equations when the cardinal number of the variables has any infinite value. The solution of this general type of equation is found [cf. ★3·32 and ★4·04]. This is effected by the help of some important definitions [cf. ★2·0, and ★2·05, and ★2·22, and ★3·10, and ★3·20]. In ★5, the application to equations with a finite number of variables is considered.

In Section II, Cantor's theory of cardinals as developed in my paper on "Cardinal Numbers" in Vol. XXIV, p. 367 of this Journal, is applied; and after determining the cardinal numbers of various classes of relations in ★10, in ★11 the number of solutions of any logical equation is determined [cf. ★11·13 and ★11·25]. In ★12, these results are considered for the special case of a finite number of variables [cf. ★12·01, and ★12·02, and ★12·2], and some examples for one and two variables are appended. In ★13, the following problem is considered: i is a given class, a and b are given classes contained in i, required the number of classes x contained in i such that the cardinal number of the class $(a \frown x) \smile (b \frown \bar{x})$, where $\bar{x}$ is the part of i not x, is some given number α. This number is determined [cf. ★13·20 to ★13·24]. Thence

[cf. ★13·30], with the same suppositions, the sum of the following series is determined:

$$\sum^{x \supset i} 2^{\mu\{(a \frown x) \smile (b \frown \breve{x})\}}.$$

These two sections are written out in the notations of Peano and Russell, explained in the memoir on "Cardinal Numbers" (loc. cit.).

Section III considers the orders of the Logical Substitution Groups, considered in my memoir on "Symbolic Logic," in Vol. XXIII, p. 297, of this Journal; the order of the complete group is $24^{\mu i}$ (cf. ★20·1); the order of the identical group of a function with invariants s_1, s_2, s_3, s_4 is

$$24^{\mu(\bar{s}_1 \smile s_4)} \times 6^{\mu\{(s_1 \frown \bar{s}_2) \smile (s_3 \frown \bar{s}_4)\}} \times 4^{\mu(s_2 \frown \bar{s}_3)}.$$

Also the orders of other groups are determined.

Section IV deals with some properties of a certain simple type of substitutions.

My memoir on "Symbolic Logic" in this Journal, Part I in Vol. XXIII, p. 140, and Part II, Vol. XXIII, p. 297, is always cited as Symb. Log., Part I or Symb. Log., Part II; the memoir "On Cardinal Numbers" in Vol. XXIV of his Journal is cited as Card. Numb.

SECTION I.

★1 $i, h \,\varepsilon\, \mathrm{cls} \,.\, P \,\varepsilon\, \mathrm{rel} \,.\, \pi \supset i \,.\, \breve{\pi} \supset h \,.\supset.\!\cdot.$

·1 $\mathrm{equ}\,(i, h, P) = \mathrm{rel} \frown R \ni [\rho = i \,.\, \breve{\rho} \supset h \,.\, \breve{R} P \supset 0']$. Df.

Note: $\breve{R} P \supset 0' .=. R \breve{P} \supset 0'$ [cf. Card. Numb., Section II, 2·13]. Here "equ" is contracted from "equation." The connection between this definition and the ordinary theory of logical equations is most easily seen from the next proposition,

·2 $\mathrm{equ}\,(i, h, P) = \mathrm{rel} \frown R \ni [\rho = i \,.\, \breve{\rho} \supset h : k \,\varepsilon\, h \,.\supset_k .\, \pi k \frown \rho k = \Lambda]$,
[★1·1 .=. Prop.]

Note: To establish the connection between these propositions and the theory of logical equations, consider h as the class of indices not necessary finite or denumerable in number: i is the class called the universe, and all the classes appearing in the equation as factors or as summands are contained in i; P is the relation determining the known

coefficients of the various terms, thus πk may be written a_k, where a_k is a known class contained in i and corresponding to the index k; since $\breve{\pi} \supset h$ and is not necessarily equal to h, it may happen that $a_k = \Lambda$; R is the relation determining the unknowns of the equation, thus: let ρk be written x_k, where x_k is a class contained in i and corresponding to the index k: then ★1·1 and the general hypothesis assert that for every product of the type $a_k \frown x_k$ we have

$$a_k \frown x_k = \Lambda,$$

and that the logical sum of all classes of the type x_k is equal to i. For example, if the number of indices is two, so that $\mu h = 2$ and these indices are 1 and 2, then

$$a_1 \frown x_1 \smile a_2 \frown x_2 = \Lambda\,, \quad x_1 \smile x_2 = i.$$

In logical equations, as ordinarily considered, we should also have $x_1 \frown x_2 = \Lambda$, so that $x_2 = \bar{x}_1$ (putting $\bar{x}_1$ for $i \sim x_1$), and the equation becomes

$$a_1 \frown x_1 \smile a_2 \frown \bar{x}_1 = \Lambda.$$

This further specialization of the general idea will be considered later; but meanwhile we shall prove a series of propositions which belong equally to the more general conception here defined.

★2 $i, h \,\varepsilon\, \text{cls} \,.\, P \,\varepsilon\, \text{rel} \,.\, \pi \supset i \,.\, \breve{\pi} \supset h \,.\supset .\cdot$

·0 $b \supset h \,.\supset .\, (b, \text{div}\, P) = i \frown x \ni (\breve{\pi} x = h \sim b).$ Df.

·01 $b \supset h \,.\, \exists\, h \sim b \sim \breve{\pi} \,.\supset .\, (b, \text{div}\, P) = \Lambda,$

$[\text{Hp} \,.\supset .\sim \exists\, i \frown x \ni (h \sim b = \breve{\pi} x) \,.\supset .\, \text{Prop}].$

·02 $\breve{\pi} = h \,.\supset .\, (\Lambda, \text{div}\, P) = x \ni (\breve{\pi}\, x = h).$

·03 $\breve{\pi} \sim = h \,.\supset .\, (\Lambda, \text{div}\, P) = \Lambda,$

$[\text{Hp} \,.\supset .\sim \exists i \frown x \ni (\breve{\pi} x = h) \,.\supset .\, \text{Prop}].$

·04 $(h, \text{div}\, P) = i \sim \pi,$

$[(h, \text{div}\, P) = i \frown x \ni (\breve{\pi} x = \Lambda) \,.\supset .\, \text{Prop}].$

·05 $(Nc, \text{div}\, P) = y \ni [\exists\, \text{cls}'h \frown b \ni \{y = (b, \text{div}\, P)\}].$ Df.

Note: "div" is contracted from "divisional": the importance of a similar conception in relation to logical equations containing a finite number of variables was exemplified by W. E. Johnson in a paper read

before the International Congress of Philosophy, Paris, 1900, in the section dealing with "Logique et Histoire des Sciences" (published by Armand Colin, Paris). The definitions, ★ 2·0 and ★ 2·05, have essential reference to i, h which are given in the general hypothesis; in other connections it might be necessary to express these classes and to write $(b, \text{div}^i_h\, P)$ for $(b, \text{div}\, P)$ and $(Nc, \text{div}^i_h\, P)$ for $(Nc, \text{div}\, P)$.

·10 $P_{\breve{\pi}} = \iota\, \text{rel} \cap P_1 \ni (\pi_1 = i : x \varepsilon i \,.\, x P_1 y \,.\supset .\, y = \breve{\pi} x)$. Df.

·11 $\breve{\pi}_{\breve{\pi}} \supset \text{cls}'h \,.\, \cup\, ' \breve{\pi}_{\breve{\pi}} = \breve{\pi}$.

·12 $P_{\breve{\pi}} \varepsilon Nc \rightarrow 1$.

·13 $b \supset h \,.\supset .\, (b, \text{div}\, P) = \pi_{\breve{\pi}} (h \sim b)$.

·20 $(Nc, \text{div}\, P) \varepsilon\, \text{cls}^2\, \text{excl}$,

$[P_{\breve{\pi}} \varepsilon Nc \rightarrow 1 \,.\supset : x P_{\breve{\pi}} y \,.\, x' P_{\breve{\pi}} y' \,.\, y \circ' y' \,.\supset .\, x \circ' x' :\supset .\, \text{Prop}]$.

·21 $\cup\, '(Nc, \text{div}\, P) = i$,

$[x \varepsilon \pi \,.\, \subset : b = h \sim \breve{\pi} x \,.\supset .\, x \varepsilon (b, \text{div}\, P) :\supset .\, x \varepsilon^2 (Nc, \text{div}\, P)$, (1)

★ $2{\cdot}04 \,.\supset : x \varepsilon i \sim \pi \,.\supset .\, x \varepsilon (h, \text{div}\, P) \,.\supset .\, x \varepsilon^2 (Nc, \text{div}\, P)$, (2)

$(1) \,.\, (2) \,.\supset .\, \text{Prop}]$.

Note: The importance of the class $(Nc, \text{div}\, P)$ depends upon ★ 2·20 and ★ 2·21.

·22 $\beta \varepsilon Nc \,.\supset .\, (\beta, \text{div}\, P) = x \ni [\exists\, \text{cls}'\, h \cap b \ni \{b \varepsilon \beta \,.\, x = (b, \text{div}\, P)\}]$. Df.

★ 3 $i, h \varepsilon\, \text{cls} \,.\, P \varepsilon\, \text{rel} \,.\, \pi \supset i \,.\, \breve{\pi} \supset h \,.\, \exists\, \text{equ}\, (i, h, P) \,.\, R \varepsilon\, \text{equ}\, (i, h, P) \,.\supset .\!\cdot.$

·01 $b \supset h \,.\, a \varepsilon (b, \text{div}\, P) \,.\, aRk \,.\supset .\, k \varepsilon b$,

$[\text{Hp} \,.\supset : a \varepsilon i \,.\, h \sim b = \breve{\pi} a \,.\supset : x \varepsilon h \sim b \,.\supset_x .\, a P x$, (1)

$(1) \,.\, x \varepsilon h \sim b \,.\, a R k \,.\, \breve{R} P \supset \circ' \,.\supset .\, x \circ' k$, (2)

$\text{Hp} \,.\, (2) \,.\supset .\, \text{Prop}]$.

·02 $(\Lambda, \text{div}\, P) = \Lambda$,

$[$★ $3{\cdot}01 \,.\, \rho = i \,.\supset : b \supset h \,.\, \exists\, (b, \text{div}\, P) \,.\supset .\, \exists\, b :\supset .\, \text{Prop}]$.

·10 $b \supset h \,.\, \exists\, (b, \text{div}\, P) \,.\supset .\, (b, \text{rel}\, P) = \text{rel} \cap S \ni [\sigma = (b, \text{div}\, P) \,.\, \breve{\sigma} \supset b]$. Df.

Note: $(b, \text{rel}\, P)$, like $(b, \text{div}\, P)$, refers essentially to i and h which are given in the general hypothesis. If it were necessary to render these classes explicit in the notation, we could write $(b, \text{rel}^i_h\, P)$ for $(b, \text{rel}\, P)$.

·11 $\beta \varepsilon Nc \,.\supset .\, (\beta, \text{rel}\, P) = x \ni [\exists\, \text{cls}'\, h \cap b \ni \{b \varepsilon \beta \,.\, x = (b, \text{rel}\, P)\}]$. Df.

·12 $(Nc, \text{rel}\, P) = x \ni [\exists\, \text{cls}'h \cap b \ni \{x = (b, \text{rel}\, P)\}]$. Df.

·13 $S \varepsilon^2 (Nc, \text{rel } P) . = . S \varepsilon \text{ rel} . \exists \text{ cls}' h \frown b \ni [\sigma = (b, \text{div } P) . \breve{\sigma} \supset b]$.

·14 $S, S' \varepsilon^2 (Nc, \text{rel } P) . S o' S' . \supset : \sigma \frown \sigma' = \Lambda . \smile . \sigma = \sigma'$,

$[\bigstar 2{\cdot}20 . \bigstar 3{\cdot}13 . \supset . \text{Prop}]$.

·20 $\{\text{rel } (Nc, \text{rel } P)^{\times}\} = \text{rel} \frown S \ni [\exists (Nc, \text{rel } P)^{\times} \frown M \ni (S = \smile {}^{\backprime} M)]$. Df.

Note: For an explanation of this use of the symbol $\times$, cf. Card. Numb. ★6·0.

·21 $S \varepsilon \{\text{rel } (Nc, \text{rel } P)^{\times}\} . \supset . \sigma = \smile {}^{\backprime}(Nc, \text{div } P) = i$.

·22 $S \varepsilon \{\text{rel } (Nc, \text{rel } P)^{\times}\} . b \supset h . \exists \ (b, \text{div } P) . \supset .$

$S_b = \iota (b, \text{rel } P) \frown T \ni [x \varepsilon b . \supset : x T y . = . x S y]$. Df.

·23 Hp $\bigstar 3{\cdot}22 . \supset . \sigma_b = (b, \text{div } P) . \breve{\sigma}_b \supset b$.

·24 $S \varepsilon \{\text{rel } (Nc, \text{rel } P)^{\times}\} . b \supset h . b' \supset h . b o' b' . \supset . \sigma_b \frown \sigma_{b'} = \Lambda$.

·25 $S \varepsilon \{\text{rel } (Nc, \text{rel } P)^{\times}\} . \supset . \breve{\sigma} \supset h$.

·26 $S \varepsilon \{\text{rel } (Nc, \text{rel } P)^{\times}\} . \supset . \breve{S} P \supset o'$,

$[x \varepsilon \breve{\sigma} . \supset . \exists \text{ cls}' h \frown b \ni \{x \varepsilon b . \exists (b, \text{div } P)\}$, (1)

$(1) . x \breve{S} z . \supset . z \varepsilon (b, \text{div } P)$, (2)

$z \varepsilon (b, \text{div } P) . z P y . \supset . y \varepsilon h \sim b$, (3)

$(1) . (2) . (3) . \supset : x \breve{S} P y . \supset . \exists \text{ cls}' h \frown b \ni \{x \varepsilon b . y \varepsilon h \sim b\} : \supset . \text{Prop}]$.

·30 $\{\text{rel } (Nc, \text{rel } P)^{\times}\} \supset \text{equ } (i, h, P)$,

$[\bigstar 3{\cdot}21 . \bigstar 3{\cdot}25 . \bigstar 3{\cdot}26 . \supset . \text{Prop}]$.

·31 $\text{equ } (i, h, P) \supset \{\text{rel } (Nc, \text{r l } P)^{\times}\}$,

$[R \varepsilon \text{ equ } (i, h, P) . \supset .\because$

$\bigstar 2{\cdot}20 . \bigstar 2{\cdot}21 . a \varepsilon i . \supset . \text{cls}' h \frown b \ni \{a \varepsilon (b, \text{div } P)\} \varepsilon 1$, (1)

$(1) . \bigstar 3{\cdot}01 . a \varepsilon i . b \varepsilon \iota \text{ cls}' h \frown b \ni \{a \varepsilon (b, \text{div } P)\} . a R k . \supset . k \varepsilon b$, (2)

$(2) . b \supset h . \exists (b, \text{div } P) . \supset . R_b \ni [\rho_b = (b, \text{div } P) . \breve{\rho}_b \supset b .\because$

$a \varepsilon (b, \text{div } P) . \supset_a : a R k . = . a R_b k] \varepsilon 1 \frown \text{cls}' (b, \text{rel } P)$, (3)

$\rho = i . \bigstar 2{\cdot}21 . (4) . \supset . R \varepsilon \{\text{rel } (Nc, \text{rel } P)^{\times}\} . \supset . \text{Prop}]$.

·32 $\text{equ } (i, h, P) = \{\text{rel } (Nc, \text{rel } P)^{\times}\}$,

$[\bigstar 3{\cdot}30 . \bigstar 3{\cdot}31 . \supset . \text{Prop}]$.

Note: ★3·32 gives the general solution for the class of relations indicated by equ (i, h, P), in the sense that $\{\text{rel } (Nc, \text{rel } P)^{\times}\}$, which has been proved to be the same class, is defined by indicating a method for the construction of any member of the class, whereas the definition of equ (i, h, P) simply indicates the general property of any member

of the class: we have here an example of two different class-concepts with the same extension.

We now proceed to specialize these ideas in the direction of ordinary logical equations.

★4 $i, h \,\varepsilon\, \text{cls} . P \,\varepsilon\, \text{rel} . \pi \supset i . \breve{\pi} \supset h . \exists\, \text{equ}\,(i, h, P) . \supset .\because$

•0 $Nc \rightarrow 1 \frown \text{equ}\,(i, h, P) = Nc \rightarrow 1 \frown R \ni \{\rho = i . \breve{\rho} \supset h . \breve{R} P \supset o'\}$.

•01 $b \supset h . \exists\, (b, \text{div}\, P) . \supset . (b, Nc \rightarrow 1, P) = Nc \rightarrow 1 \frown (b, \text{rel}\, P)$. Df.

•02 $(Nc, Nc \rightarrow 1, P) = x \ni [\exists\, \text{cls}' h \frown b \ni \{x = (b, Nc \rightarrow 1, P)\}]$. Df.

•03 $\{\text{rel}\,(Nc, Nc \rightarrow 1, P)^{\times}\}$

$$= \text{rel} \frown S \ni [\exists\, (Nc, Nc \rightarrow 1, P)^{\times} \frown M \ni \{S = \frown 'M\}]. \quad \text{Df.}$$

•04 $Nc \rightarrow 1 \frown \text{equ}\,(i, h, P) = \{\text{rel}\,(Nc, Nc \rightarrow 1, P)^{\times}\}$,
[★3•32 . $\supset$. Prop].

Note: With the notation of the note on ★1•2, we have, if $R \,\varepsilon\, Nc \rightarrow 1 \frown \text{equ}\,(\iota, h, P)$,

$$a_k \frown x_k = \Lambda,$$

and the logical sum of all the classes of the type x_k is equal to i, and the logical product of any two different classes of the type x_k, say x_k and $x_{k'}$, where k is different from k', is nu l, that is,

$$k, k' \,\varepsilon\, h . k\, o'\, k' . \supset . x_k \frown x_{k'} = \Lambda.$$

Thus the class of classes of the type x_k is exhaustive of i and the classes are mutually exclusive. For instance, if the number of indices is 2, so that $h \,\varepsilon\, 2$, and if $\bar{x}_1$ is put for $i \sim x_1$, then $x_1 \smile x_2 = i$ and $x_1 \frown x_2 = \Lambda$; hence $x_2 = \bar{x}_1$, and the equation becomes

$$a_1 \frown x_1 \smile a_2 \frown \bar{x}_1 = \Lambda.$$

The general relation of the above theorems to logical equations with a finite number of unknowns is considered in the next set of propositions ★5•0 to ★5•11; and equations with a finite number of variables are again considered in set ★12.

We shall use the following notation wherever the symbol i represents a class

$x \supset i . \supset_x . \bar{x} = i \sim x$.

★5 $i \,\varepsilon\, \text{cls} : x \supset i . \supset_x . \bar{x} = i \sim x : \nu \,\varepsilon\, Nc \text{ fin} . h = Nc \frown \beta \ni (o < \beta \leqq \nu) :$
$P \,\varepsilon\, \text{rel} . \pi \supset i . \breve{\pi} \supset h . \exists\, \text{equ}\,(i, h, P) : \beta \,\varepsilon\, h . \supset . a_\beta = \pi \beta :$
$R \,\varepsilon\, \text{equ}\,(i, h, P) . \beta \,\varepsilon\, h . \supset . x_\beta = \rho \beta : \supset .\because$

•0 $\lambda \, \varepsilon \, Nc \,.\, 0 < \lambda \leqq \nu \,.\, b = \iota 1 \cup \iota 2 \cup \iota 3 \cup \ldots\ldots \iota\lambda \,.\supset .$
$(b, \text{ div } P) = \bar{a}_1 \frown \bar{a}_2 \frown \ldots\ldots \frown \bar{a}_\lambda \frown a_{\lambda+1} \frown a_{\lambda+2} \frown \ldots\ldots \frown a_\nu$
$[\bigstar 2{\cdot}0 \,.\supset .\, \text{Prop}]$.

•01 $(\Lambda, \text{ div } P) = a_1 \frown a_2 \frown \ldots\ldots \frown a_\nu = \Lambda,$
$[\bigstar 2{\cdot}0 \,.\, \bigstar 3{\cdot}02 \,.\, \text{Hp} \,.\supset .\, \text{Prop}]$.

.02 $(h, \text{ div } P) = \bar{a}_1 \frown \bar{a}_2 \frown \ldots\ldots \frown \bar{a}_\nu$.

•03 $\beta \, \varepsilon \, h \,.\, A = Z \ni [\exists \, h \frown \lambda \ni (Z = a_\lambda)] \,.$
$D^A_\beta = y \ni [\exists \; C^A_\beta \frown u \ni (y = \frown {}^\backprime u)] \,.\supset .\, S_\beta = \cup {}^\backprime D^A_\beta.$ Df

Note: S_1, S_2, ... S_ν are the symmetric functions of a_1, a_2, ..., a. as defined in "Symb. Logic," Part I, §2; thus, $S_1 = a_1 \cup a_2 \cup \ldots \cup a_\nu$ and $S_\nu = a_1 \frown a_2 \frown \ldots \frown a_\nu$.

•04 $S_0 = i$. Df.

Note: This definition is convenient to preserve the generality of certain formulæ.

•05 $\beta \, \varepsilon \, Nc \,.\, \beta \leqq \nu \,.\supset .\, \cup {}^\backprime (\beta, \text{ div } P) = S_{\nu-\beta} \frown \bar{S}_{\nu-\beta+1},$
$[\bigstar 2{\cdot}22 \,.\, \bigstar 5{\cdot}0 \,.\, \bigstar 5{\cdot}03 \,.\supset .\, \text{Prop}]$.

•06 $S_\nu = \Lambda,$ $[\bigstar 5{\cdot}01 \,.=.\, \text{Prop}]$.

•10 $R \, \varepsilon \, \text{equ } (i, h, P) \,.\supset :\, (a_1 \frown x_1) \cup (a_2 \frown x_2) \cup \ldots\ldots \cup (a_\nu \frown x_\nu) = \Lambda \,.$
$x_1 \cup x_2 \cup \ldots\ldots \cup x_\nu = i,$
$[\text{Hp} \, (\bigstar 5) \,.\supset .\, \text{Prop}]$.

•11 $R \, \varepsilon \, Nc \rightarrow 1 \frown \text{equ} \, (i, h, P) \,.\supset .\!\cdot .\, \bigstar 5{\cdot}10 : \lambda, \lambda' \, \varepsilon \, h \,.$
$\lambda o' \lambda' \,.\supset .\, x_\lambda \frown x_{\lambda'} = \Lambda.$

Note: Comparing this with the ordinary type of logical equation, for instance, in two variables,

$$(a \frown x \frown y) \cup (b \frown x \frown \bar{y}) \cup (c \frown \bar{x} \frown y) \cup (d \frown \bar{x} \frown \bar{y}) = \Lambda,$$

we see that $x_1 = x \frown y$, $x_2 = x \frown \bar{y}$, $x_3 = \bar{x} \frown y$, $x_4 = \bar{x} \frown \bar{y}$. Thus $x = x_1 \cup x_2$, $y = x_1 \cup x_3$. Also for the comparison to hold, ν must be a number of the type 2^δ, and then δ is the number of unknowns in the ordinary logical equation. But whatever ν may be, the equation

$$(a_1 \frown x_1) \cup (a_2 \frown x_2) \cup \ldots\ldots \cup (a_\nu \frown x_\nu) = \Lambda,$$

where $x_1 \cup x_2 \cup \ldots\ldots \cup x_\nu = i$ and $x_\lambda \frown x_{\lambda'} = \Lambda$, $(\lambda o' \lambda')$ can always be modified into an equation of the required type.

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For, let $\delta \leqq \nu \leqq 2^\delta$, and let $a_{\nu+1}, a_{\nu+2}, \ldots a_{2^\delta}$ be each equal to i, so that

$$\nu < \lambda \leqq 2^\delta . \supset . a_\lambda = i,$$

then $$\nu < \lambda \leqq 2^\delta . x_\lambda \supset i . a_\lambda \frown x_\lambda = \Lambda . \supset . x_\lambda = \Lambda .$$

Hence by adding on to $a_1, \ldots a_\nu$ the $(2^\delta - \nu)$ terms $a_{\nu+1}, \ldots a_{2^\delta}$ (all equal to i), and to $x_1, \ldots x_\nu$ the $(2^\delta - \nu)$ terms $x_{\nu+1}, \ldots x_{2^\delta}$ (all equal to Λ), we obtain

$$(a_1 \frown x_1) \smile (a_2 \frown x_2) \smile \ldots \smile (a_{2^\delta} \frown x_{2^\delta}) = \Lambda,$$

where $x_1 \smile x_2 \smile \ldots \smile x_{2^\delta} = i$ and $x_\lambda \frown x_{\lambda'} = \Lambda$, $(\lambda o' \lambda')$, and $x_1, x_2, \ldots x_\nu$ can be any set of terms satisfying the unmodified equation and can be no other set. Hence there is no loss of generality in supposing that ν is always of the form 2^δ. The next set of propositions (★6) will deal with the generalization of this reasoning for the case when ν may be infinite.

★6 $i, h \,\varepsilon\, \text{cls} . \nu \,\varepsilon\, Nc . h \,\varepsilon\, \nu . R \,\varepsilon\, Nc \rightarrow 1 . \rho = i . \breve{\rho} \supset h . \supset .\because$

·10 $\exists\, Nc \frown \delta \ni (\delta \leqq \nu \leqq 2^\delta)$.

·2 $P \,\varepsilon\, \text{rel} . \pi \supset i . \breve{\pi} \supset h . \exists\, \text{equ}\,(i, h, P) . \delta \,\varepsilon\, Nc . \delta \leqq \nu \leqq 2^\delta .$
$h' \,\varepsilon\, \text{cls} . h \frown h' = \Lambda . h \smile h' = 2^\delta .$
$P' = \imath\, \text{rel} \frown P'' \ni [\pi'' \supset i . \breve{\pi}'' \supset h \smile h' .\because z \,\varepsilon\, h . \supset : x P' z . = . x P z .\because$
$z \,\varepsilon\, h' . x \,\varepsilon\, i . \supset_{x, z} . x P' z] . \supset : \text{equ}\,(i, h \smile h', P') = \text{equ}\,(i, h, P)$.

Note: This proposition, of which the proof is easy, shows that there is no loss of generality in always assuming, when convenient, ν to be of the form 2^δ.

SECTION II.

The Cardinal Numbers of Various Classes.

★10 $u, v \,\varepsilon\, \text{cls} . u \frown v = \Lambda . \supset .\because$

·01 $(\supset u, \text{rel}, \supset v) = \text{rel} \frown R \ni (\rho \supset u . \breve{\rho} \supset v)$. Df.

·11 $(u, \text{rel}, \supset v) = \text{rel} \frown R \ni (\rho = u . \breve{\rho} \supset v)$. Df.

·12 $(\supset u, \text{rel}, v) = \text{rel} \frown R \ni (\rho \supset u . \breve{\rho} = v)$. Df.

·13 $(u, \text{rel}, v) = \text{rel} \frown R \ni (\rho = u . \breve{\rho} = v)$. Df.

·2 $(u\,; v) = (x, y) \ni (x \,\varepsilon\, u . y \,\varepsilon\, v)$. Df.

·30 $\mu(u;v) = \mu u \times \mu v$, [cf. Card. Numb. ★ 7·21].

·31 $\mu(\supset u, \text{rel}, \supset v) = \mu\, \text{cls'}(u;\, v) = 2^{\mu u \times \mu v}$, [cf. Card. Numb. ★ 15·0].

·32 $\mu(u, \text{rel}, \supset v) = (2^{\mu v} - 1)^{\mu u}$,

[$x \varepsilon u . \supset . k_x = l \ni \{l \supset \iota x \cup \text{cls'}\, v \sim \iota\Lambda . x \varepsilon l . l \cap \text{cls'}\, v \varepsilon 1\} .\because$

$k = p \ni \{\exists\, u \cap x \ni (p = k_x)\} : \supset . k \varepsilon \mu u . k \supset \mu\, \text{cls'}\, v \sim \iota\Lambda$, (1)

$m \varepsilon k^{\times} . \supset .\because x \varepsilon u . \supset_x . m \cap k_x \varepsilon 1 : m \supset u \cup \text{cls'}\, v \sim \iota\Lambda . u \supset m :: \supset .$

$\mu k^{\times} = \mu(u, \text{rel}, \supset v)$, (2)

(Card. Numb. ★ 12·1) . (1) . (2) . $\supset$. Prop].

·33 $\mu u + \mu v \varepsilon Nc$ infin $. \mu u > 1 . \mu v > 1 . \supset .$

$\mu(u, \text{rel}, \supset v) = 2^{\mu u \times \mu v} = \mu(\supset u, \text{rel}, v)$.

·40 $Nc \Rightarrow 1 \cap (u, \text{rel}, \supset v) = v^{u}$, [cf. Card. Numb. ★ 14·0].

·41 $\mu\{Nc \Rightarrow 1 \cap (u, \text{rel}, \supset v)\} = \mu v^{\mu u}$, [cf. Card. Numb. ★ 14·1].

·51 $\mu\{Nc \Rightarrow 1 \cap (\supset u, \text{rel}, \supset v\} = (1 + \mu v)^{\mu u}$,

[$w \varepsilon C^{u}_{\beta} . \supset . \mu\{Nc \Rightarrow 1 \cap (w, \text{rel}, \supset v)\} = (\mu v)^{\beta}$, (1)

$(1) . \supset . \mu\{Nc \Rightarrow 1 \cap (\supset u, \text{rel}, \supset v)\} = \sum^{\beta \leqq \mu u} C^{\mu u}_{\beta} \times (\mu v)^{\beta}$, (2)

(Card. Numb. ★ 17·4) . (2) . $\supset$. Prop].

.52 $\mu\{1 \Rightarrow Nc \cap (\supset u, \text{rel}, \supset v)\} = (1 + \mu u)^{\mu v}$.

·61 $\mu u > 1 . \mu v > 1 . \supset : (2^{\mu v - 1} - 1)^{\mu u - 1} \leqq \mu(u, \text{rel}, v) \leqq (2^{\mu v} - 1)^{\mu u}$.

$(2^{\mu u - 1} - 1)^{\mu v - 1} \leqq \mu(u, \text{rel}, v) \leqq (2^{\mu u} - 1)^{\mu v}$,

[$x \varepsilon u . y \varepsilon v . R \varepsilon (u \sim \iota x, \text{rel}, \supset v \sim \iota y) . R' \varepsilon \text{rel} . \rho' = \iota x .$

$\breve{\rho}' = v \sim \breve{\rho}(u \sim \iota x) . \supset . \exists\, \breve{\rho}' . \supset . R \cup R' \varepsilon (u, \text{rel}, v)$, (1)

$(1) . x \varepsilon u . y \varepsilon v . \supset . \mu(u, \text{rel}, v) \geqq \mu(u \sim \iota x, \text{rel}, \supset v \sim \iota y)$, (2)

(2) . ★ 10·32 . $(u, \text{rel}, v) \supset (u, \text{rel}, \supset v) . \supset$. Prop].

·62 $\mu u > 1 . \mu v > 1 . \mu u + \mu v \varepsilon Nc$ infin $. \supset . \mu(u, \text{rel}, v) = 2^{\mu u \times \mu v}$,

[★ 10·61 . $\supset$. Prop].

★ 11 $i, h \varepsilon \text{cls} . P \varepsilon \text{rel} . \pi \supset i . \breve{\pi} \supset h . \exists\, \text{equ}(i, h, P) . \supset .\because$

·0 $\mu\, \text{equ}(i, h, P) = \mu\{\text{rel}(Nc, \text{rel } P)^{\times}\} = \mu\{Nc, \text{rel } P\}^{\times}$.

[★ 3·32 . ★ 3·20 . $\supset$. Prop].

·01 $\mu\, \text{equ}(i, h, P) = \prod^{\beta \leqq \mu h} \mu(\beta \text{ rel } P)^{\times}$,

[(Card. Numb. ★ 10·22) . ★ 11·0 . ★ 3·11 . ★ 3·12 . $\supset$. Prop].

·11 $b \supset h . \supset . \mu(b, \text{rel } P) = (2^{\mu \breve{b}} - 1)^{\mu(b, \text{div} P)}$,

[★ 3·10 . ★ 10·32 . $\supset$. Prop].

·12 $\beta \mathrel{\varepsilon} Nc \,.\, \supset . \, \mu\,(\beta, \text{rel } P)^{\times} = (2^{\beta} - 1)^{\mu \smile {}'(\beta, \text{div } P)}$,
[★3·11 . ★2·22 . ★11·11 . (Card. Numb. ★10·22 . ★13·1) . $\supset$. Prop].

·13 $\mu \text{ equ}\,(i, h, P) = \prod^{\beta \leqq \mu h} (2^{\beta} - 1)^{\mu \smile (\beta, \text{div } P)}$,
[★11·01 . ★11·12 . $\supset$. Prop].

Note: This is the general formula for the number of relations belonging to the class equ (i, h, P), and thus also for the number of solutions of the corresponding logical equation.

·21 $\mu\{Nc \to 1 \frown \text{equ}(i, h, P)\} = \mu\{\text{rel}\,(Nc, Nc \to 1, P)^{\times}\} = \mu\{Nc, Nc \to 1, P\}^{\times}$,
[★4·03 . ★4·04 . $\supset$. Prop].

·22 $\mu\,\{Nc \to 1 \frown \text{equ}\,(i, h, P)\} = \prod^{\beta \leqq \mu h} \mu\,(\beta, Nc \to 1, P)^{\times}$,
[(Card. Numb. ★10·22) . $\supset$. Prop].

·23 $b \supset h \,.\, \supset . \, \mu\,(b, Nc \to 1, P) = (\mu\, b)^{\mu\,(b, \text{div } P)}$,
[★4·01 . ★3·10 . ★10·41 . $\supset$. Prop].

·24 $\beta \mathrel{\varepsilon} Nc \,.\, \supset . \, \mu\,(\beta, Nc \to 1, P)^{\times} = \beta^{\smile {}'\mu\,(\beta, \text{div } P)}$,
[★3·11 . ★2·22 . (Card. Numb. ★10·22 ★13·1) . $\supset$. Prop].

·25 $\mu\,\{Nc \to 1 \frown \text{equ}\,(i, h, P)\} = \prod^{\beta \leqq \mu h} \beta^{\mu \smile {}'(\beta, \text{div } P)}$,
[★11·22 . ★11·24 . $\supset$. Prop].

Note: This is the general formula for the number of relations belonging to the class $Nc \to 1 \frown \text{equ}\,(i, h, P)$, and thus also for the number of solutions of the corresponding logical equation. M. Poretsky has given the number of solutions of a logical equation in one variable (viz., $a \frown x \smile b \frown \bar{x} = \Lambda$) in the Revue de Mathématiques, Turin, Tome VI, 1896, in his paper, "La Loi des racines en Logique." The solution given now holds for any finite or infinite number of variables. We proceed to state the propositions ★11·13 and ★11·25 in forms convenient for the case where the number of variables in the logical equations is finite; this case has already been partially considered in ★5.

★12 Hp (★5) . $\supset$.·.

·01 $\mu \text{ equ}\,(i, h, P) = \prod_{\beta > \nu}^{\beta \leqq \nu} (2^{\beta} - 1)^{\mu\,(S_{\nu-\beta} \frown \bar{S}_{\nu-\beta+1})}$,
[★5·05 . ★11·13 . $\supset$. Prop].

$$\cdot 02 \quad \mu \{ Nc \rightarrow 1 \quad \text{equ}\,(i,\, h,\, P) \} = \prod_{\beta > 1}^{\beta \leqq \nu} \beta^{\mu (S_{\nu-\beta} \quad \bar{S}_{\nu-\beta+1})},$$

[★ 5·05 . ★ 11·25 . ⊃ . Prop].

Note: ★ 12·01 and ★ 12·02 give the number of solutions of the two types of logical equation when the number of variables is finite; ★ 12·02 is of fundamental importance, especially in the theory of Logical Substitution Groups, developed in Section III. It can be verified (the number of variables being finite) by another method.

$$\cdot 03 \quad \mu \bar{S}_{\nu-1} \,\varepsilon\, Nc \text{ infin} \,.\, \supset .\; \mu \,\text{equ}\,(i,\, h,\, P) = \mu \{ Nc \rightarrow 1 \frown \text{equ}\,(i,\, h,\, P) \} = 2^{\mu \bar{S}_{\nu-1}}$$

$[\alpha_2, \ldots . \; \alpha_\nu \,\varepsilon\, Nc \,.\, \alpha_2 + \ldots . + \alpha_\nu \,\varepsilon\, Nc \text{ infin} \,.\, \supset .$

$$\prod_{\beta > 1}^{\beta \leqq \nu} (2^\beta - 1)^{\alpha_\beta} = \prod_{\beta > 1}^{\beta \leqq \nu} \beta^{\alpha_\beta} = 2^{\Sigma \alpha_\beta} \qquad (1)$$

$$\begin{aligned} 1 < \beta \leqq \nu \,.\, \supset .\; & \bar{S}_{\nu-\beta} \smile (S_{\nu-\beta} \frown \bar{S}_{\nu-\beta+1}) \\ & = \bar{S}_{\nu-\beta} \smile (\bar{S}_{\nu-\beta} \frown \bar{S}_{\nu-\beta+1}) \smile (S_{\nu-\beta} \frown \bar{S}_{\nu-\beta+1}) \\ & = \bar{S}_{\nu-\beta} \smile \bar{S}_{\nu-\beta+1} = \bar{S}_{\nu-\beta+1}. \qquad (2) \end{aligned}$$

$$\begin{aligned} (2) \,.\, S_0 = i \,.\, \supset .\; & \mu (S_{\nu-2} \frown \bar{S}_{\nu-1}) + \mu (S_{\nu-3} \frown \bar{S}_{\nu-2}) + \ldots . \\ & + \mu (S_0 \frown \bar{S}_1) = \mu \bar{S}_{\nu-1}, \qquad (4) \end{aligned}$$

Hp . (1) . (4) . ★ 12·01 . ★ 12·02 . ⊃ . Prop].

Note: This proposition is a great simplification of ★ 12·01 and of ★ 12·02 in the most important case.

$$\cdot 1 \quad \mu \,\text{equ}\,(i,\, h,\, P) \,\varepsilon\, Nc \text{ fin} \,.\, \mu \{ Nc \rightarrow 1 \frown \text{equ}\,(i,\, h,\, P) \} \,\varepsilon\, Nc \text{ fin} : \smile :$$

$\mu \bar{S}_{\nu-1} \,\varepsilon\, Nc$ infin,

[Demonst (★ 12·03) . ⊃ . Prop].

Note: It follows from ★ 12·03 and ★ 12·1 that the number of solutions of a logical equation is either finite or is a number not less than that of the continuum.

$$\cdot 2 \quad 2^{\mu \bar{S}_{\nu-1}} \leqq \mu \{ Nc \rightarrow 1 \frown \text{equ}\,(i,\, h,\, P) \} \leqq \nu^{\mu \bar{S}_{\nu-1}},$$

[★ 12·02 . (demonstration of ★ 12·03) . ⊃ . Prop].

Examples. (A) of ★ 12·02,

$$(a \frown x) \smile (b \frown \bar{x}) = \Lambda.$$

Here $\nu = 2$, $S_1 = a \smile b$, $S_2 = a \frown b = \Lambda$; hence the number of solu-

tions is $2^{\mu \bar{S}_1} = 2^{\mu(\bar{a} \frown \bar{b})}$. This example is the case given by Poretsky.

(B) of ★12·02,

$$(a \frown x \frown y) \smile (b \frown x \frown \bar{y}) \smile (c \frown \bar{x} \frown y) \smile (d \frown \bar{x} \frown \bar{y}) = \Lambda .$$

Here $\nu = 4$, $S_1 = a \smile b \smile c \smile d, \ldots ,$ $S_\nu = a \frown b \frown c \frown d = \Lambda$; hence the number of its solutions is

$$2^{\mu(S_2 \frown \bar{S}_3)} \times 3^{\mu(S_1 \frown \bar{S}_2)} \times 4^{\mu \bar{S}} = 2^{2 \times \mu S_1 + \mu(S_2 \frown \bar{S}_3)} \times 3^{\mu(S_1 \frown \bar{S}_2)}$$

and if $\mu \bar{S}_3$ is infinite, it follows from ★12·03 that the number of solutions can be written in the simplified form $2^{\mu \bar{S}_3}$, where

$$\bar{S}_3 = (\bar{a} \frown \bar{b}) \smile (\bar{a} \frown \bar{c}) \smile (\bar{a} \frown \bar{d}) \smile (\bar{b} \frown \bar{c}) \smile (\bar{b} \frown \bar{d}) \smile (\bar{c} \frown \bar{d}).$$

(C) of ★12·01,

$$(a \frown x_1) \smile (b \frown x_2) = \Lambda, \quad x_1 \smile x_2 = i.$$

Here $\nu = 2$, $S_1 = a \smile b$, $S_2 = a \frown b = \Lambda$; and the number of solutions is $(2^2 - 1)^{\mu \bar{S}_1} = 3^{\mu(\bar{a} \frown \bar{b})}$.

(D) of ★12·01,

$$(a \frown x_1) \smile (b \frown x_2) \smile (c \frown x_3) \smile (d \frown x_4) = \Lambda, \quad x_1 \smile x_2 \smile x_3 \smile x = \Lambda.$$[4]

Here $\nu = 4$, $S_1 = a \smile b \smile c \smile d, \ldots ,$ $S_\nu = a \frown b \frown c \frown d = \Lambda$; the number of solutions is

$$(2^2 - 1)^{\mu(S_2 \frown \bar{S}_3)} \times (2^3 - 1)^{\mu(S_1 \quad \bar{S}_2)} \times (2^4 - 1)^{\mu \bar{S}_1}.$$

Since, in ★12·02, the coefficients of the equation only enter into the answer through the invariants $S_1, \ldots S_\nu$, it follows that all equations whose left-hand sides are members of the same congruent family (cf. Symb. Log., Part II, §6), have the same number of solutions; for instance, considering an equation with two unknowns, such as that in example (B) above, for the family of secondary linear primes (cf. Symb. Log., Part I, §3), $S_1 = i$, $S_2 = i$, $S_3 = i$, $S_4 = \Lambda$; hence the number of solutions is 1, as is otherwise known (cf. Symb. Log., Part I, §3). For the family of secondary separable primes, $S_1 = i$, $S_2 = \Lambda$, $S_3 = \Lambda$, $S_4 = \Lambda$; hence the number of solutions is $3^{\mu i}$. This can be verified by considering the equation $x \frown y = \Lambda$. For the family of deficiency two and of supplemental deficiency two, $S_1 = i$, $S_2 = i$, $S_3 = \Lambda$, $S_4 = \Lambda$; and hence the number of solutions is $2^{\mu i}$. This is

immediately obvious from considering the equation (in two variables), $x \frown (y \smile \bar{y}) = \Lambda$, that is, $x = \Lambda$, and y can be any class subordinate to i.

★13 $i \,\varepsilon\, \mathrm{cls} : x \supset i \,.\, \supset_x .\, \bar{x} = i \sim x : \supset \therefore$

·01 $a, b \,\varepsilon\, \mathrm{cls}' i \,.\, u = z \ni [\exists\, \mathrm{cls}' i \frown x \ni \{z = (a \frown x) \smile (b \frown \bar{x})\}] \,.\, \supset :$

$$\mu u = 2^{\mu(\bar{a} - b)},$$

$[\mu u = \mu\, \mathrm{cls}' (\bar{a} \frown b) \,.\,$ (Card. Numb. ★15·0) $.\, \supset .\,$ Prop].

·11 $a \,\varepsilon\, \mathrm{cls}' i \,.\, \beta \,\varepsilon\, Nc \,.\, \beta \leqq \mu a \,.\, u = \mathrm{cls}' i \frown x \ni (x \frown a \,\varepsilon\, C^i_\beta) \,.\, \supset .\, \mu u = C^{\mu a}_\beta \times 2^{\mu \bar{a}},$

$[p = \iota C^a_\beta \smile \iota\, \mathrm{cls}' \bar{a} \,.\, \supset .\, u = x \ni [\exists p^\times \frown m \ni (x = \smile {}^{\backprime} m)] \,.\, \supset .\,$ Prop].

·20 $a, b \,\varepsilon\, \mathrm{cls}' i \,.\, a \frown b = \Lambda \,.\, \alpha \,\varepsilon\, Nc \,.\, \alpha \leqq \mu (a \smile b) \,.$

$u = \mathrm{cls}' i \frown x \ni [(a \frown x) \smile (b \frown \bar{x}) \,\varepsilon\, C^i_\alpha] \,.\, \supset .\, \mu u = C^{\mu(a \smile b)}_\alpha \times 2^{\mu p(a, \bar{b})}.$

Note: $p(a, b) = (a \frown \bar{b}) \smile (\bar{a} \smile b) \,.\, p(a, \bar{b}) = (a \frown b) \smile (\bar{a} \frown b)$ (cf. Symb. Log., Part I, §3). The proof is as follows:

$[\mathrm{Hyp} \,.\, \supset .\, u = y \ni [\exists\, Nc \frown (\xi, \eta) \ni \{\breve{\xi} + \eta = \alpha \,.\, \exists\, (x_1, x_2, u) \ni (x_1 \,\varepsilon\, C^a_\xi \,.$
$x_2 \,\varepsilon\, C^b_\eta \,.\, u \supset i \,.\, y = x_1 \smile \bar{x}_2 \smile u \frown p(a, \bar{b}))\}]. \qquad (1)$

$(1) \,.\,$ (Card. Numb. ★7·21) $.\, \supset .\, \mu u = \sum_{\breve{\xi} + \eta = \alpha} C^{\mu a}_\xi \times C^{\mu b}_\eta \times 2^{\mu p(a, b)}, \qquad (2)$

$(2) \,.\,$ (Card. Numb. ★16·1) $.\, \supset .\,$ Prop].

·21 $a, b \,\varepsilon\, \mathrm{cls}' i \,.\, \alpha \,\varepsilon\, Nc\, \mathrm{fin} \smile N\alpha_0 \,.\, \mu(a \frown b) < \alpha \leqq \mu(a \smile b) \,.$

$u = \mathrm{cls}' i \frown x \ni [(a \frown x) \smile (b \frown \bar{x}) \,\varepsilon\, C^i_\alpha] \,.\, \supset .\, \mu u = C^{\mu p(a, b)}_{\alpha - \mu(a \frown b)} \times 2^{\mu p(\bar{a}, b)},$

$[(a \frown x) \smile (b \frown \bar{x}) \,\varepsilon\, C^i_\alpha \,. = .\, (a \frown \bar{b} \frown x) \smile (\bar{a} \frown b \frown \bar{x}) \,\varepsilon\, C^i_{\alpha - \mu(a \frown b)}, \qquad (1)$

$(1) \,.\,$ ★13·20 $.\, \supset .\,$ Prop].

For the definition of $N\alpha_0$, cf. Card. Numb. ★30·0. The point of the limitation to Nc fin or to $N\alpha_0$ is that then $\alpha - \mu(a \frown b)$ is a definite number.

·22 $a, b \,\varepsilon\, \mathrm{cls}' i \,.\, \alpha \,\varepsilon\, N\alpha_0 \,.\, \mu(a \frown b) < \alpha \leqq \mu(a \smile b) \,.$

$u = \mathrm{cls}' i \frown x \ni [(a \frown x) \smile (b \frown \bar{x}) \,\varepsilon\, C^i_\alpha] \,.\, \supset .$

$\mu u = C^{\mu p(a, b)}_\alpha \times 2^{\mu p(a, \bar{b})},$

$[\alpha - \mu(a \frown b) = \alpha \,.\,$ ★13.21 $.\, \supset .\,$ Prop].

·23 $a, b \,\varepsilon\, \mathrm{cls}' \iota \,.\, \alpha \,\varepsilon\, Nc\, \mathrm{fin} \,.\, \mu(a \frown b) = \alpha \,.$

$u = \mathrm{cls}' i \frown x \ni [(a \frown x) \smile (b \frown \bar{x}) \,\varepsilon\, C^i_\alpha] \,.\, \supset .\, \mu u = 2^{\mu p(a, \bar{b})},$

[★12·02 $.\, u = \mathrm{cls}' i \frown x \ni [(a \frown b \frown x) \smile (\bar{a} \frown b \frown \bar{x}) = \Lambda] \,.\, \supset .\,$ Prop].

$$\cdot 24 \quad a,\, b\, \varepsilon\, \mathrm{cls}'\, i \,.\, \alpha\, \varepsilon\, N\alpha_0 \,.\, \mu\,(a \quad b) = \alpha \,.$$
$$u = \mathrm{cls}'\, i \cap x \ni [(a \cap x) \cap (b \cup \bar{x})\, \varepsilon\, C^i_a] \,.\supset .$$
$$\mu\, u = \sum^{\nu \leqq \alpha} C_\nu^{\mu\, p\,(a,\, b)} \times 2^{\mu\, p\,(a,\, \bar{b})},$$
$$[\nu \leqq \alpha \,.\supset .\, \alpha + \nu = \alpha \,.\supset .\, u = \mathrm{cls}'\, i \cap x \ni [\exists\, Nc \cap \nu \ni \{\nu \leqq \alpha \,.$$
$$(a \cap \bar{b} \cap x) \cup (\bar{a} \cap b \cap \bar{x})\, \varepsilon\, C^i_\nu\}], \tag{1}$$
$$\bigstar 13{\cdot}20 \,.\, (1) \,.\supset .\, \mathrm{Prop}] .$$

$$\cdot 30 \quad a,\, b\, \varepsilon\, \mathrm{cls}'\, i \,.\, a \cap b = \Lambda \,.\supset .\, \sum^{x\,)\,i} 2^{\mu\{(a \frown x) \smile (b \frown \breve{x})\}} = 2^{\mu\, p\,(a,\, \bar{b})} \times 3^{\mu\, p\,(a,\, b)},$$
$$[\bigstar 13{\cdot}20 \,.\supset .\, \sum^{x\,)\,i} 2^{\mu\{(a \frown x) \smile (b \frown \breve{x})\}} = \sum^{\breve{\xi} \leqq \mu\,(a \smile b)} C_{\breve{\xi}}^{\mu\,(a \smile b)} \times 2^{\mu\, p\,(a,\, \bar{b})} \times 2^{\breve{\xi}}$$
$$= 2^{\mu\, p\,(a,\, \bar{b})} \times \sum^{\breve{\xi} \leqq \mu\,(a \smile b)} C_{\breve{\xi}}^{\mu\,(a \smile b)} \times 2^{\breve{\xi}}, \tag{1}$$
$$(1) \,.\, (\text{Card. Numb. } \bigstar 17{\cdot}4) \,.\, a \cup b = p\,(a,\, b) \,.\supset .\, \mathrm{Prop}] .$$

$$\cdot 31 \quad a,\, b\, \varepsilon\, \mathrm{cls}'\, i \,.\supset .\, \sum^{x\,)\,i} 2^{\mu\{(a \frown x) \smile (b \frown \breve{x})\}} = 2^{\mu\,(a \frown b) + \mu\, p\,(\bar{a},\, b)} \times 3^{\mu\, p\,(a,\, b)},$$
$$[2^{\mu\{(a \quad x) \smile (b \frown \breve{x})\}} = 2^{\mu\,(a \frown b)} \times 2^{\mu\{(a \frown \bar{b} \frown \bar{x}) \quad (a \frown b \frown \breve{x})\}}, \tag{1}$$
$$p\,(a \cap \bar{b},\, \bar{a} \cap b) = p\,(a,\, b) \,.\, p\,\{a \cap \bar{b},\, (\bar{a} \cap b)\} = p\,(a,\, \bar{b}), \tag{2}$$
$$\bigstar 13{\cdot}30 \,.\, (1) \,.\, (2) \,.\supset .\, \mathrm{Prop}] .$$

SECTION III.

Orders of Various Logical Substitution Groups.

The properties of these groups have been investigated (cf. Symb. Log., Part II) for the case of functions of two variables. In the present section the orders of the various groups, discussed in the memoir referred to, will be determined. In the reference the theory of substitution groups is not investigated by symbolic methods, accordingly, these methods will be largely abandoned in the present section. A group is a class of operations and the order of the group is the cardinal number of the class. The class i will be assumed to contain the classes denoted by the two variables x and y, and also the classes denoted by the coefficients of any function of one or both of these variables. Also as before,

$$z \supset i \,.\supset_z .\, z = i \sim z.$$

The statement that $\left.\begin{matrix}\breve{\xi}_1, \breve{\xi}_2, \breve{\xi}_3, \breve{\xi}_4\\ \eta_1, \eta_2, \eta_3, \eta_4\end{matrix}\right\}$ are the coefficients of a substitution T, means that

$$Tx = (\breve{\xi}_1 \cap x \cap y) \cup (\breve{\xi}_2 \cap x \cap \bar{y}) \cup (\breve{\xi}_3 \cap \bar{x} \cap y) \cup (\breve{\xi}_4 \cap \bar{x} \cap \bar{y}),$$
$$Ty = (\eta_1 \cap x \cap y) \cup (\eta_2 \cap x \cap \bar{y}) \cup (\eta_3 \cap \bar{x} \cap y) \cup (\eta_4 \cap \bar{x} \cap \bar{y}),$$

where $\left.\begin{matrix}\breve{\xi}_1, \ldots \breve{\xi}_4\\ \eta_1, \ldots \eta_4\end{matrix}\right\}$ satisfy the equation (cf. Symb. Log., Part II, §2, equation (12)).

★ 20·01 $\Sigma\{(\breve{\xi}_p \cap \breve{\xi}_q) \cup (\bar{\bar{\breve{\xi}}}_p \cap \bar{\bar{\breve{\xi}}}_q)\} \cap \{(\eta_p \cap \eta_q) \cup (\bar{\eta}_p \cap \bar{\eta}_q)\} = \Lambda,$
$(p, q = 1, 2, 3, 4),$

which can also be written in the form

·02 $\Sigma\, \bar{p}(\breve{\xi}_p, \breve{\xi}_q) \cap \bar{p}(\eta_p, \eta_q) = \Lambda,\ (p, q = 1, 2, 3, 4),$

and also in the form

·03 $\Pi(\bar{\bar{\breve{\xi}}}_r \cup \bar{\eta}_r) \cup \Pi(\bar{\bar{\breve{\xi}}}_r \cup \eta_r) \cup \Pi(\breve{\xi}_r \cup \bar{\eta}_r) \cup \Pi(\breve{\xi}_r \cup \eta_r) = \Lambda,\ (r = 1, 2, 3, 4),$

and also in the form

·04 $\Pi\{(\bar{\bar{\breve{\xi}}}_r \cap \eta_r) \cup (\bar{\bar{\breve{\xi}}}_r \cap \bar{\eta}_r) \cup (\breve{\xi}_r \cap \bar{\eta}_r)\} \cup \Pi\{(\bar{\bar{\breve{\xi}}}_r \cap \eta_r) \cup (\bar{\bar{\breve{\xi}}}_r \cap \bar{\eta}_r) \cup (\breve{\xi}_r \cap \eta_r)\}$
$\cup\, \Pi\{(\breve{\xi}_r \cap \eta_r) \cup (\breve{\xi}_r \cap \bar{\eta}_r) \cup (\bar{\bar{\breve{\xi}}}_r \cap \bar{\eta}_r)\}$
$\cup\, \Pi\{(\breve{\xi}_r \cap \eta_r) \cup (\breve{\xi}_r \cap \bar{\eta}_r) \cup (\bar{\bar{\breve{\xi}}}_r \cap \eta_r)\} = \Lambda.$

Any one of these forms will be called the equation of condition for the coefficients of a substitution. When this equation of condition is fully developed in terms of its eight unknowns $\breve{\xi}_1 \ldots \breve{\xi}_4, \eta_1 \ldots \eta_4$, it has 2^8 terms; the coefficients of these various terms are either i or Λ. Considering the form ★ 20·04, it is easily seen that the first product (i. e., $\Pi\{(\bar{\bar{\breve{\xi}}}_r \cap \eta_r) \cup (\bar{\bar{\breve{\xi}}}_r \cap \bar{\eta}_r) \cup (\breve{\xi}_r \cap \bar{\eta}_r)\}$) gives 3^4 terms with coefficient i; the second product gives $(3^4 - 2^4)$ other terms with coefficient i, the third product gives $(3^4 - 2^4 - 2^4 + 1)$ other terms with coefficient i, the fourth product gives $(3^4 - 2^4 - 2^4 - 2^4 + 3)$ other terms with coefficient i. Hence, there are 232 terms with coefficient i and 24 with with coefficient Λ.

Thus, calculating the invariants $S_0, \ldots S_{2^8}$ of this equation from S_0 to S_{2^8-24}, they are each equal to i, and from S_{2^8-23} to S_{2^8} they are each equal to Λ. Hence from ★ 12·02 we deduce

·1 The order of the complete logical substitution group for functions of two variables is $24^{\mu i}$. Thus, if the order is infinite, it is equal to the power of the continuum at least.

The identical group of $\phi(x, y)$ is simply isomorphic with that of the canonical function of the congruent family (cf. Symb. Log., Part II, §7) to which $\phi(x, y)$ belongs. Hence, in order to determine the order of the identical group of $\phi(x, y)$, we have only to determine it for the congruent family. Let s_1, s_2, s_3, s_4 be the invariants of this family, so that $s_4 \supset s_3 \supset s_2 \supset s_1$. Then the coefficients of a substitution of the identical group of the canonical function, in addition to satisfying the equation of condition (★ 20·04) must satisfy (cf. Symb. Log., Part II, §7, equ. (37)), the four equations

$$\left.\begin{array}{l}
(s_1 \cap \bar{s}_2 \cap \breve{\xi}_1 \cap \bar{\eta}_1) \cup (s_1 \cap \bar{s}_3 \cap \breve{\breve{\xi}}_1 \cap \eta_1) \cup (s_1 \cap \bar{s}_4 \cap \breve{\breve{\breve{\xi}}}_1 \cap \bar{\eta}_1) = \Lambda, \\
(s_1 \cap \bar{s}_2 \cap \breve{\xi}_2 \cap \eta_2) \cup (s_2 \cap \bar{s}_3 \cap \breve{\breve{\xi}}_2 \cap \eta_2) \cup (s_2 \cap \bar{s}_4 \cap \breve{\breve{\breve{\xi}}}_2 \cap \bar{\eta}_2) = \Lambda, \\
(s_1 \cap \bar{s}_3 \cap \breve{\xi}_3 \cap \eta_3) \cup (s_2 \cap \bar{s}_3 \cap \breve{\breve{\xi}}_3 \cap \bar{\eta}_3) \cup (s_3 \cap \bar{s}_4 \cap \breve{\breve{\breve{\xi}}}_3 \cap \bar{\eta}_3) = \Lambda, \\
(s_1 \cap \bar{s}_4 \cap \breve{\xi}_4 \cap \eta_4) \cup (s_2 \cap \bar{s}_4 \cap \breve{\breve{\xi}}_4 \cap \bar{\eta}_4) \cup (s_3 \cap \bar{s}_4 \cap \breve{\breve{\breve{\xi}}}_4 \cap \eta_4) = \Lambda.
\end{array}\right\} (1)$$

The equations can be combined with the equation of condition into one single equation of condition with 2^8 terms when fully developed. By noticing the symmetry of equations (1) and that of ★ 20·04, and reducing by the relations between s_1, s_2, s_3, s_4, we find that the values of the 2^8 coefficients are given by the following table, the values of the coefficients being on the upper line and the corresponding number on the lower line being the number of coefficients with that value:

i	Λ	$s_1 \cap \bar{s}_4$	$s_1 \cap \bar{s}_3$	$s_2 \cap \bar{s}_4$	$s_2 \cap \bar{s}_3$	$s_3 \cap \bar{s}_4$	$s_1 \cap \bar{s}_2$	$(s_1 \cap \bar{s}_2) \cup (s_3 \cup \bar{s}_4)$
232,	1,	13,	3,	3,	1,	1,	1,	1

Hence the values of the invariants of the equation $S_0, S_1, \ldots\ldots S_{2^8}$ can be calculated. After some reduction we find that from $\beta = 0$ to $\beta = 2^8 - 24$ inclusive, $S_\beta = i$; and that from $\beta = 2^8 - 23$ to $\beta = 2^8 - 6$, inclusive, $S_\beta = s_1 \cap \bar{s}_4$; and that for $\beta = 2^8 - 5$ and $\beta = 2^8 - 4$, $S_\beta = s_2 \cap \bar{s}_3$; and that from $\beta = 2^8 - 3$ to $\beta = 2^8$, inclusive, $S_\beta = \Lambda$. Hence, from ★ 12·02, we deduce:

·2 The order of the identical group of any member of the congruent family (s_1, s_2, s_3, s_4), where $s_4 \supset s_3 \supset s_2 \supset s_1$ is

$$24^{\mu(\bar{s}_1 \frown s_4)} \times 6^{\mu\{(s_1 \frown \bar{s}_2) \smile (s_3 \frown \bar{s}_4)\}} \times 4^{\mu(s_2 \frown \bar{s}_3)},$$

which can also be written

$$12^{\mu(\bar{s}_1 \smile s_4)} \times 3^{\mu\{(s_1 \frown \bar{s}_2) \smile (s_3 \frown \bar{s}_4)\}} \times 2^{\mu(s_2 \frown \bar{s}_3)} \times 2^{\mu i}.$$

And from ★ 12·03, or from ★ 20·2, we deduce

★ 20·21 If i is an infinite class, the order of the identical group of any function is $2^{\mu i}$.

Examples. The order of the identical group of a linear secondary prime [of congruent family (i, i, i, Λ)] is $6^{\mu i}$; this is also the order for a separable secondary prime [of congruent family $(i, \Lambda, \Lambda, \Lambda)$].

The order of the identical group of a function of deficiency two and of supplemental deficiency two [of congruent family (i, i, Λ, Λ)] is $4^{\mu i}$.

If $\phi(x, y)$ and $\psi(x, y)$ be any two functions of x and y, we shall always denote by $\Theta(\phi, \psi)$ a certain important function of their coefficients, defined as follows:

Let

$$\phi(x, y) = (a_1 \frown x \frown y) \smile (a_2 \frown x \frown \bar{y}) \smile (a_3 \frown \bar{x} \frown y) \smile (a_4 \frown \bar{x} \frown \bar{y}),$$

and

$$\psi(x, y) = (b_1 \frown x \frown y) \smile (b_2 \frown x \frown \bar{y}) \smile (b_3 \frown \bar{x} \frown y) \smile (b_4 \frown \bar{x} \frown \bar{y}),$$

then $\Theta(\phi, \psi) = \Sigma\, \bar{p}(a_r, a_s;\, b_r, b_s), \quad (r, s = 1, 2, 3, 4).$

Thus $\Theta(\phi, \psi) = \Theta(\psi, \phi).$

Also $\Theta(\phi, \psi)$ is the same as the left-hand side of ★ 20·02, after substituting a for $\breve{\xi}$ and b for η; hence, after the same substitution, $\Theta(\phi, \psi)$ can be written in the form of the left-hand side of ★ 20·01, or of ★ 20·03, or of ★ 20·04. Also $\Theta(\phi, \psi) = \Lambda$ is the condition that $\phi(x, y)$ and $\psi(x, y)$ should be a pair of director-functions (cf. Symb. Log., Part II, §1) of some substitution.

We shall now prove the following theorem:

·22 If $\phi(x, y)$ and $\psi(x, y)$ are two functions of x and y, such that $\mu\Theta(\phi, \psi)$ is infinite, the order of the common subgroup of the identical groups of $\phi(x, y)$ and of $\psi(x, y)$ is $2^{\mu\Theta(\phi, \psi)}$; and otherwise the order of the subgroup is finite.

For (cf. Symb. Log., Part II, §8, equ. (37)) the coefficients $\left.\begin{matrix}\breve{\xi}_1, \breve{\xi}_2, \breve{\xi}_3, \breve{\xi}_4 \\ \eta_1, \eta_2, \eta_3, \eta_4\end{matrix}\right\}$, of any substitution of the common subgroup, satisfy in

addition to ★ 20·04, the four equations

$$\{p\,(a_2,\,a_1;\;b_2,\,b_1)\frown\breve{\xi}_1\frown\bar{\eta}_1\}\smile\{p\,(a_3,\,a_1;\;b_3,\,b_1)\frown\bar{\breve{\xi}}_1\frown\eta_1\}$$
$$\smile\{p\,(a_4,\,a_1;\;b_4,\,b_1)\frown\bar{\breve{\xi}}_1\frown\bar{\eta}_1\}=\Lambda,$$
$$\{p\,(a_1,\,a_2;\;b_1,\,b_2)\frown\breve{\xi}_2\frown\eta_2\}\smile\{p\,(a_3,\,a_2;\;b_3,\,b_2)\frown\bar{\breve{\xi}}_2\frown\eta_2\}$$
$$\smile\{p\,(a_4,\,a_2;\;b_4,\,b_2)\frown\bar{\breve{\xi}}_2\frown\bar{\eta}_2\}=\Lambda,$$
$$\{p\,(a_1,\,a_3;\;b_2,\,b_3)\frown\breve{\xi}_3\frown\eta_3\}\smile\{p\,(a_2,\,a_3;\;b_2,\,b_3)\frown\breve{\xi}_3\frown\bar{\eta}_3\}$$
$$\smile\{p\,(a_4,\,a_3;\;b_4,\,b_3)\frown\bar{\breve{\xi}}_3\frown\bar{\eta}_3\}=\Lambda,$$
$$\{p\,(a_1,\,a_4;\;b_1,\,b_4)\frown\breve{\xi}_4\frown\eta_4\}\smile\{p\,(a_2,\,a_4;\;b_2,\,b_4)\frown\breve{\xi}_4\frown\bar{\eta}_4\}$$
$$\smile\{p\,(a_3,\,a_4;\;b_3,\,b_4)\frown\bar{\breve{\xi}}_4\frown\eta_4\}=\Lambda.$$

Thus the complete condition, satisfied by the coefficients, is an equation in eight variables $\breve{\xi}_1 \ldots\ldots \breve{\xi}_4,\ \eta_1 \ldots\ldots \eta_4$ with 2^8 terms, of which $2^8 - 24$ are i, one is Λ, and the remaining 23 are equal either to single coefficients of the four equations above, or to sums of these coefficients. Now if $d_1,\ d_2 \ldots\ldots d_{23}$ are these remaining coefficients and $S_0,\ S_1, \ldots\ldots S_{2^8}$ are the invariants of the equation, it is easy to see that

$$\bar{S}_{2^8-1}=\bar{d}_1\smile\bar{d}_2\smile\ldots\ldots\smile\bar{d}_{23},$$

and hence

$$\bar{S}_{2^8-1}=\bar{p}\,(a_1,\,a_2;\;b_1,\,b_2)\smile\bar{p}\,(a_1,\,a_3;\;b_1,\,b_3)\smile\bar{p}\,(a_1,\,a_4;\;b_1,\,b_4)$$
$$\smile\bar{p}\,(a_2,\,a_3;\;b_2,\,b_3)\smile\bar{p}\,(a_2,\,a_4;\;b_2,\,b_4)\smile\bar{p}\,(a_3,\,a_4;\;b_3,\,b_4)=\Theta\,(\theta,\,\psi).$$

Accordingly, the required proposition follows from ★ 12·03.

Note: I have not succeeded in shortening the labor of calculating the 256 invariants $S_0,\ S_1, \ldots\ldots S_{2^8}$ of the complete equation satisfied by the coefficients of any substitution of the common subgroup of the identical groups of two functions. Accordingly, I have not deduced the order, when finite, of this common subgroup. But from ★ 12·2, we deduce:

★ 20·3 The order of the common subgroup of the identical groups of and two functions $\phi\,(x,\,y)$ and $\psi\,(x,\,y)$ is not less than $2^{\mu\,\Theta\,(\phi,\,\psi)}$ and not greater than $2^{8\times\mu\,\Theta\,(\phi,\,\psi)}$.

From this we deduce, as a corollary, that if the order of this common subgroup is unity, the functions $\phi\,(x,\,y)$ and $\psi\,(x,\,y)$ are a pair of director-functions of some substitution, a proposition already known (cf. Symb. Log., Part II, §8).

★ 21·1 The cyclical group generated by any substitution T is, in general, of the 12^{th} order.

For, let

$$Tx = (a_1 \frown x \frown y) \smile (a_2 \frown x \frown \bar{y}) \smile (a_3 \frown \bar{x} \frown y) \smile (a_4 \frown \bar{x} \frown \bar{y}),$$
$$Ty = (b_1 \frown x \frown y) \smile (b_2 \frown x \frown \bar{y}) \smile (b_3 \frown \bar{x} \frown y) \smile (b_4 \frown \bar{x} \frown \bar{y}).$$

Since (cf. Symb. Log., Part II, §2),

$$a_p \frown a_q \frown a_r = \Lambda = \bar{a}_p \frown \bar{a}_q \frown \bar{a}_r, \qquad (p, q, r \text{ unequal}),$$

and $\quad a_p \frown a_q = a_r \frown a_s, \qquad (p, q, r, s \text{ unequal}),$

it follows that in the complete development of i in terms of a_1, a_2, a_3, a_4 (2^4 terms), the only terms not vanishing can be written in the form $a_p \frown a_q$. Similarly for b_1, b_2, b_3, b_4.

Let $\quad A_{pq} = a_p \frown a_q, \quad B_{pq} = b_p \frown b_q, \qquad (p, q = 1, 2, 3, 4).$

Then, from the condition for a substitution,

$$A_{pq} \frown B_{pq} = \Lambda, \quad A_{pq} \frown B_{rs} = \Lambda,$$

where, as in the sequel, different subscripts are unequal. Also

$$a_1 = A_{12} \smile A_{13} \smile A_{14}, \quad b_1 = B_{12} \smile B_{13} \smile B_{14},$$
$$\bar{a}_1 = A_{23} \smile A_{34} \smile A_{42}, \quad \bar{b}_1 = B_{23} \smile B_{34} \smile B_{42},$$

with similar equations for other subscripts.

Also put X_1 for $x \frown y$, X_2 for $x \frown \bar{y}$, X_3 for $\bar{x} \frown y$, X_4 for $\bar{x} \frown \bar{y}$. Then

$$X_p \frown X_q = \Lambda.$$

Hence
$$\begin{aligned} Tx &= \Sigma A_{pq} \frown (X_p \smile X_q), && (p, q = 1, 2, 3, 4), \\ Ty &= \Sigma B_{pq} \frown (X_p \smile X_q), && \text{“} \qquad \text{“} \\ T\bar{x} &= \Sigma A_{pq} \frown (X_r \smile X_s), && (p, q, r, s = 1, 2, 3, 4), \\ T\bar{y} &= \Sigma B_{pq} \frown (X_r \smile X_s), && \text{“} \qquad \text{“} \end{aligned}$$

and
$$\begin{aligned} TX_1 &= Tx \frown Ty = \Sigma A_{pq} \frown B_{pr} \frown X_p, && \text{“} \qquad \text{“} \\ TX_2 &= Tx \frown T\bar{y} = \Sigma A_{pq} \frown B_{pr} \frown X_q, && \text{“} \qquad \text{“} \\ TX_3 &= T\bar{x} \frown Ty = \Sigma A_{pq} \frown B_{pr} \frown X_r, && \text{“} \qquad \text{“} \\ TX_4 &= T\bar{x} \frown T\bar{y} = \Sigma A_{pq} \frown B_{pr} \frown X_s, && \text{“} \qquad \text{“} \end{aligned}$$

Thus $T\{A_{pq} \frown B_{qr} \frown X_1\} = A_{pq} \frown B_{qr} \frown X_p$, $(p, q, r, s = 1, 2, 3, 4)$,
$T\{A_{pq} \frown B_{qr} \frown X_2\} = A_{pq} \frown B_{qr} \frown X_p$, " "
$T\{A_{pq} \frown B_{qr} \frown X_3\} = A_{pq} \frown B_{qr} \frown X_r$, " "
$T\{A_{pq} \frown B_{qr} \frown X_4\} = A_{pq} \frown B_{qr} \frown X_s$, " "

Hence if P is one of the terms
$A_{pq} \frown B_{qr} \frown X_p$, or $A_{pq} \frown B_{qr} \frown X_q$, or $A_{pq} \frown B_{qr} \frown X_r$, or $A_{pq} \frown B_{qr} \frown X_s$, we can easily verify that, either $TP = P$, or $T^2P = P$, or $T^3P = P$, or $T^4P = P$; for instance,

$$T^3\{A_{23} \frown A_{24} \frown X_1\} = T^2\{A_{23} \frown A_{24} \frown X_3\}$$
$$= T\{A_{23} \frown A_{24} \frown X_4\} = A_{23} \frown A_{24} \frown X_1,$$
$$T^4\{A_{23} \frown A_{24} \frown X_3\} = T^3\{A_{23} \frown A_{24} \frown X_4\}$$
$$= T^2\{A_{23} \frown A_{24} \frown X_1\} = T\{A_{23} \frown A_{24} \frown X_2\} = A_{23} \frown A_{24} \frown X_3.$$

Hence the smallest number n for which the equation $T^nP = P$ holds for every term P of the type defined above is 12.

But remembering the conditions satisfied by a_1, a_2, a_3, a_4 b_1, b_2, b_3, b_4, we see that we can write

$$\phi(x, y) = \Sigma g \frown A_{pq} \frown B_{qr} \frown X,$$

where $p, q, r = 1, 2, 3, 4$, g is any coefficient and X is any one of X_1, X_2, X_3, X_4. Hence the proposition follows.

SECTION IV.

The Group of Primary Prime Substitutions.

Consider a substitution T such that Tx and Ty are each functions of one variable only, not the same for both; for instance, we will suppose that Tx is a function of x only, and Ty is a function of y only. Then, if $\left.\begin{matrix}\breve{\xi}_1, \breve{\xi}_2, \breve{\xi}_3, \breve{\xi}_4 \\ \eta_1, \eta_2, \eta_3, \eta_4\end{matrix}\right\}$ are the coefficients of T, we must have

$$\breve{\xi}_1 = \breve{\xi}_2,\ \breve{\xi}_3 = \breve{\xi}_4;\quad \eta_1 = \eta_3,\ \eta_2 = \eta_4,$$

and hence from ★20·01, $\breve{\xi}_3 = \bar{\breve{\xi}}_1$, $\eta_2 = \bar{\eta}_1$.

Thus, $Tx = (\bar{\breve{\xi}} \frown x) \smile (\breve{\xi} \frown \bar{x}) = p(\breve{\xi}, x)$, $Ty = p(\eta, y)$,

is the general form for such a substitution; both Tx and Ty are pri-

mary primes. Let such a substitution be called a primary prime substitution.

★ 22·0 Substitutions of the type $Tx = p(\breve{\xi}, x)$, $Ty = p(\eta, y)$ form an abelian group, in which every substitution is of the second order.

For if T' be the substitution, $T'x = p(\breve{\xi}', x)$, $T'y = p(\eta', y)$, then

$$T'Tx = \{\bar{p}(\breve{\xi}, \breve{\xi}') \frown x\} \smile \{p(\breve{\xi}, \breve{\xi}') \frown \bar{x}\},$$
$$T'Ty = \{\bar{p}(\eta, \eta') \frown y\} \smile \{p(\eta, \eta') \frown \bar{y}\}.$$

Hence $T'T$ is a substitution of the same form.

Further, these equations show that

$$TT' = T'T \text{ and } T^2 = T_0,$$

where T_0 is the identical substitution. Hence the group is abelian, and every primary prime substitution is of order two.

★ 22·1 The order of the complete group of primary prime substitutions is $4^{\mu i}$.

For whatever classes contained in i, $\breve{\xi}$ and η may be $Tx = p(\breve{\xi}, x)$ $Ty = p(\eta, y)$ belongs to the group.

·2 The class of congruent families (s_1, s_2, s_3, s_4) such that if $\phi(x, y)$ and $\psi(x, y)$ are members of the same family of the class, a primary prime substitution T can be found such that $T\phi(x, y) = \psi(x, y)$, is the class of congruent families for which $s_2 = s_3$.

For if a_1, a_2, a_3, a_4 are the coefficients of $\phi(x, y)$ and b_1, b_2, b_3, b_4 of $\psi(x, y)$, and $\breve{\xi}$, η are the parameters of the required substitution, then (cf. Symb. Log., Part II, §6, equ (31)) the condition for these two functions is

$$\begin{aligned}
&[\{p(a_1, b_1) \smile p(a_2, b_2) \smile p(a_3, b_3) \smile p(a_4, b_4)\} \frown \breve{\xi} \frown \eta] \smile \\
&\quad [\{p(a_2, b_1) \smile p(a_1, b_2) \smile p(a_4, b_3) \smile p(a_3, b_4)\} \frown \breve{\xi} \frown \bar{\eta}] \smile \\
&\quad [\{p(a_3, b_1) \smile p(a_4, b_2) \smile p(a_1, b_3) \smile p(a_2, b_4)\} \frown \bar{\breve{\xi}} \frown \eta] \smile \\
&\quad [\{p(a_4, b_1) \smile p(a_3, b_2) \smile p(a_2, b_3) \smile p(a_1, b_4)\} \frown \bar{\breve{\xi}} \frown \bar{\eta}] = \Lambda.
\end{aligned}$$

Now we know that the functions must be congruent, hence all we have to do is to seek the condition that any function $\phi(x, y)$ can be so transformed into the canonical function of its family; hence we may put

$$b_1 = s_1, \quad b_2 = s_2, \quad b_3 = s_3, \quad b_4 = s_4,$$

where s_1, s_2, s_3, s_4 are the invariants of the family and $s_4 \supset s_3 \supset s_2 \supset s_1$.

Then the resultant of the above equation, i. e., the condition for its possibility reduces to

$$(a_1 \frown a_3 \frown \bar{a}_2 \frown \bar{a}_4) \smile (a_2 \frown a_3 \frown \bar{a}_1 \frown \bar{a}_4)$$
$$\smile (a_1 \frown a_4 \frown \bar{a}_2 \frown \bar{a}_3) \smile (a_2 \frown a_4 \frown \bar{a}_1 \frown \bar{a}_3) = \Lambda .$$

Hence, remembering that functions of the same family exist with a_1, a_2, a_3, a_4 interchanged, we find $s_2 \frown \bar{s}_3 = \Lambda$, that is, $s_2 \supset s_3$. But $s_3 \supset s_2$, hence $s_2 = s_3$.

We notice that the families (i, i, i, Λ) and $(i, \Lambda, \Lambda, \Lambda)$ both belong to this class of families.

★ 22·3 The class of congruent families (s_1, s_2, s_3, s_4), such that $s_2 = s_3$, is such that if $\phi(x, y)$ and $\psi(x. y)$ be any two members of the same family, a substitution T can be found such that

$$T\phi(x, y) = \psi(x, y) \quad \text{and} \quad T\psi(x, y) = \phi(x, y).$$

This follows from ★ 22·0 and ★ 22·2.

·4 The identical group of any function of the family (s_1, s_2, s_3, s_4) contains a primary prime subgroup of order

$$2^{\mu(s_2 \frown \bar{s}_3)} \times 4^{\mu(\bar{s}_1 \smile s_4)}.$$

For in the demonstration of ★ 22·2 make $\phi(x, y)$ and $\psi(x, y)$ identical by putting a_1, a_2, a_3, a_4 for b_1, b_2, b_3, b_4, then the parameters, $\breve{\xi}$ and η, of the required primary prime substitution must satisfy

$$[\{p(a_1, a_2) \smile p(a_3, a_4)\} \frown \breve{\xi} \frown \bar{\eta}] \smile [\{p(a_1, a_3) \smile p(a_2, a_4)\} \frown \bar{\bar{\xi}} \frown \eta] \smile$$
$$[\{p(a_1, a_4) \smile p(a_2, a_3)\} \frown \xi \frown \bar{\eta}] = \Lambda.$$

This equation is always possible, and if S_1, S_2, S_3, S_4 are its invariants, we find

$$S_2 \frown \breve{S}_3 = s_2 \frown \bar{s}_3, \quad S_1 \frown \bar{S}_2 = \Lambda, \quad \bar{S}_1 = \bar{s}_1 \smile s_4 .$$

Hence from ★ 12·02 the proposition follows.

July 4, 1901.